Numerical Method for Laplace's Equation on a Rectangular Region

Junseo Lee McLean Milavickas
University of Illinois at Chicago

MSCS Undergraduate Research Laboratory

UIC

Summary

In this project, we have dealt with Laplace's Equation to study the temperature distribution on a 10×5 meter rectangular plate with its walls subject to constant temperatures (Dirichlet Boundary Conditions) as well as insulated boundaries (Neumann Boundary Conditions).

Keywords: Laplace's Equation, Central Difference Method, Dirichlet Boundary Conditions, Neumann Boundary Conditions

Motivation

Second order partial differential equations model a plethora of physical processes and phenomena. Solving Laplace's equation has not only applications in thermal physics, but in electrostatics to model potential through a finite region, or in fluid mechanics to idealize flow.

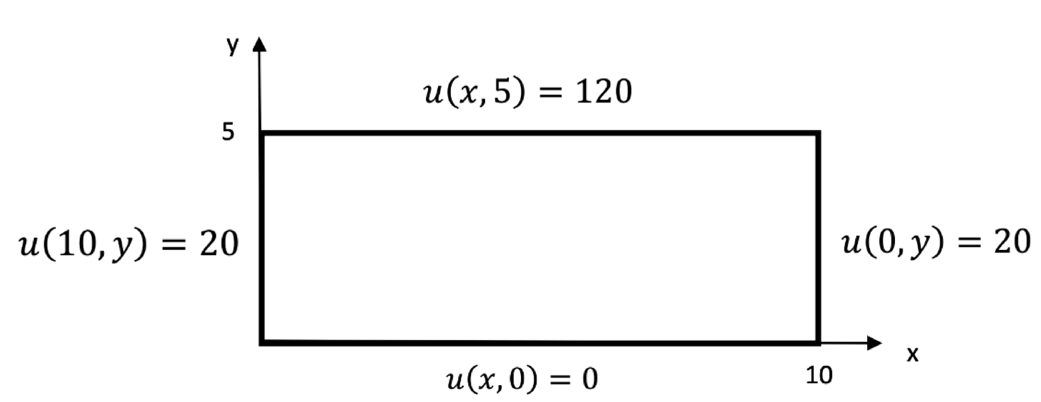
Statement of the Problem

In the study of heat conduction, Laplace's Equation describes steady-state two dimensional heat transfer. The solution to the equation, u, gives the temperature distribution of a rectangular region that does not change with time.

Laplace's Equation states

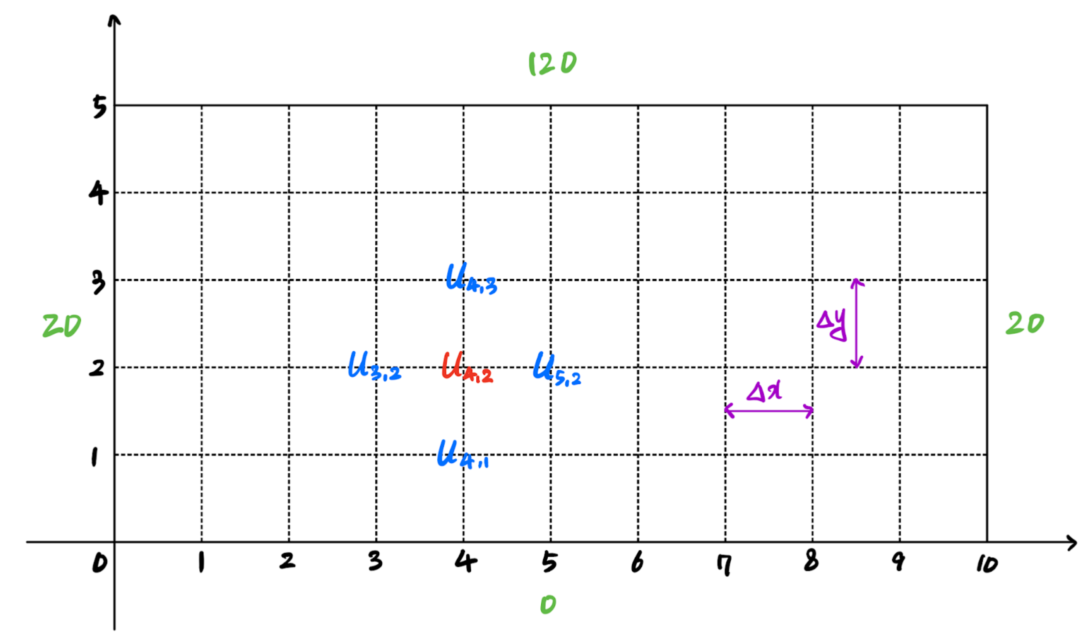
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The edges of the rectangular region are first subject to Dirichlet Boundary Conditions. This means that each edge is held at a constant temperature. Based on the wall temperature, the solution $u(x,y)$ to Laplace's Equation in the domain is required to satisfy the boundaries shown below:



How It Works - Direchlet Boundary

In the Laplace's Equation, the forward and backward approximations extend outside the boundaries and cannot be used. So the second-order central difference method can be used to approximate the partial derivatives.



- Set $\Delta x = \Delta y$, and then make the grid.
- We know the Boundary Conditions, so set them
- Use "guess value" of interior grid
- Iterate the equation until the difference between value before and after the iteration is small enough using Central Finite Difference Method

Finite-Difference Approximations

There are three ways to discretize Laplace's Equation:

- Second-Order Forward Finite Difference
$$u_{i,j} = \frac{2u_{i+1,j} + 2u_{i,j+1} - u_{i+2,j} - u_{i,j+2}}{2}$$
- Second-Order Backward Finite Difference
$$u_{i,j} = \frac{2u_{i-1,j} + 2u_{i,j-1} - u_{i-2,j} - u_{i,j-2}}{2}$$
- Second-Order Central Finite Difference
$$u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}}{4}$$

Math

Starting with Δu, we must discretize each partial derivative in the x and y directions.

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,}}{(\Delta x)^2}$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2}$$

Adding the two together gives

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} + \left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} = 0$$

Math

Since $\Delta x = \Delta y$, the denominators can be eliminated

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0$$

Solving for $u_{i,j}$ gives

$$u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}}{4}$$

Thus, the central difference method approximation for the two dimensional heat equation is the average of the points surrounding the point being calculated.

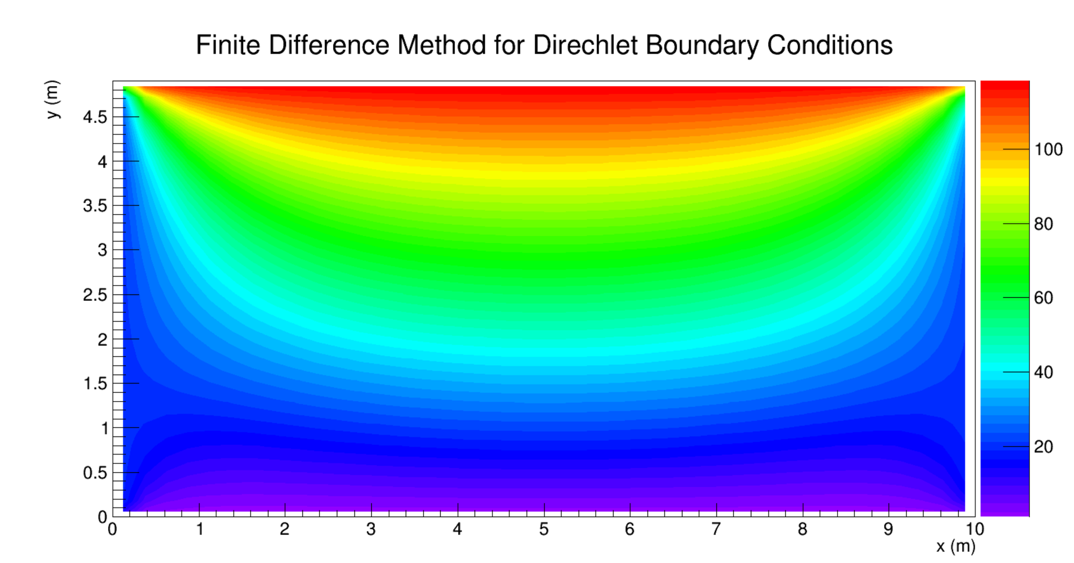
The numerical method can be compared to an exact analytical solution to the Direchlet problem.

$$\sum_{n=1}^{\infty} A_n \sin(\mu_n x)\sinh(\mu_n y) + \sum_{m=1}^{\infty} B_m \sinh(\lambda_m x)\sin(\lambda_m y)$$

$$\mu_n = \frac{n\pi}{a}, \lambda_m = \frac{m\pi}{b}, A_n = \frac{480}{n\pi\sinh(\frac{n\pi b}{a})}, n = 1,3,5...$$

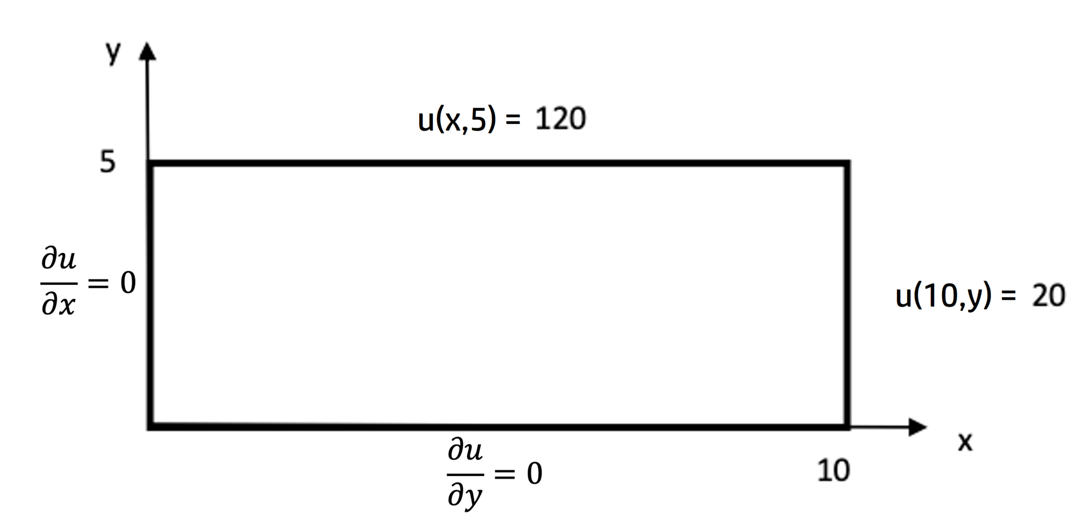
$$\text{and } B_m = \frac{160}{n\pi\sinh(\frac{n\pi a}{b})}, m = 1,3,5...$$

Numerican Solution- Direchlet Boundary Conditions



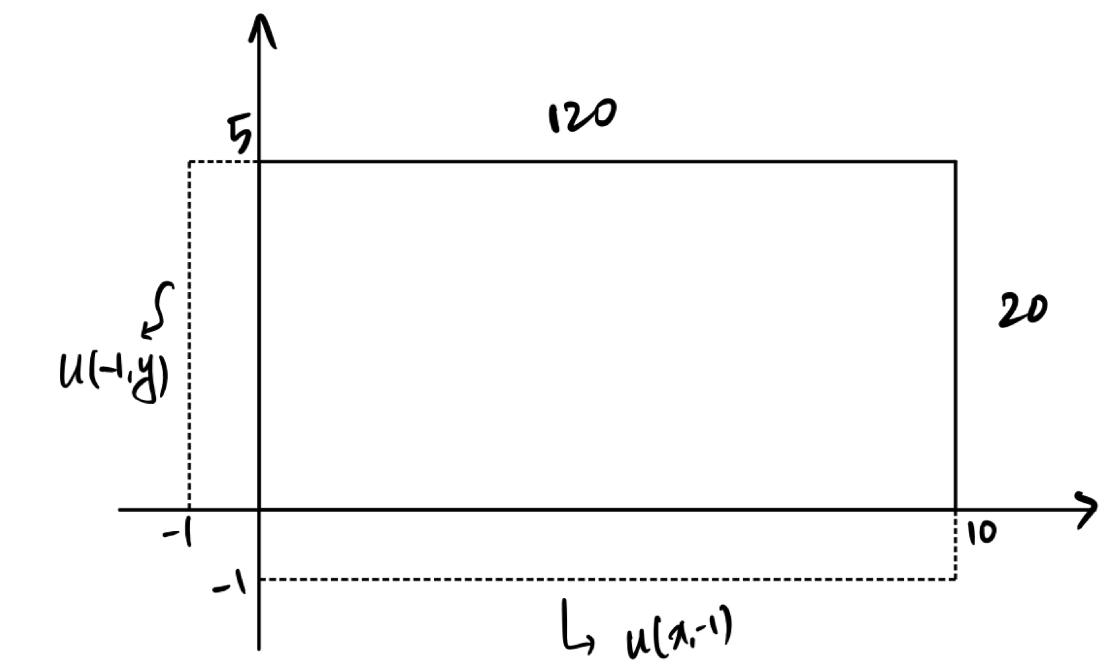
Laplace's Equation with Neumann BC

We also solved a Laplace's Equation with Neumann Boundary Conditions for $x = 0$ and $y = 0$.



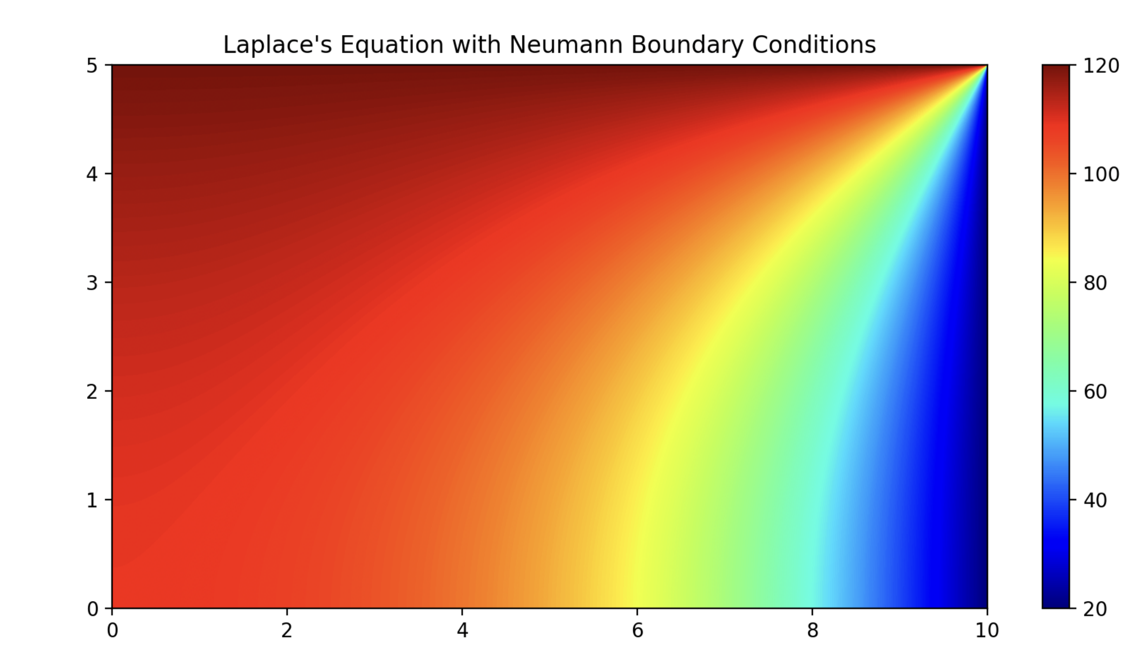
How it works - on a Neumann BC

To figure out the values at $x = 0$ and $y = 0$, we set negative ghost boundaries outside the physical domain to express the derivative at the boundary.



- We got the equation $u_{-1,y} = u_{1,y}$ and $u_{x,-1} = u_{x,1}$ as $u_y(x,-1) = u_x(-1,y) = 0$.
- Using the central difference method, calculate the values at $x = 0$ and $y = 0$
- Represent the complete discretized equation for the problem as a matrix

Result - on a Neumann BC



Conclusions

- Laplace's equation describes two dimensional heat transfer
- The central difference method for second order derivatives is used to approximate the partial derivatives.
- If $\Delta x = \Delta y$, then each point in the grid is simply the average of the points surrounding it.

References

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R. Nagle, E. Saff, A. Snider. *Fundamentals of Differential Equations*. Pearson (2019).