Computational Nonimaging Geometric Optics: Monge-Ampere

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Summary

For this research, we took a look at how the law of reflection and the conservation of energy work when defining the shape of a mirror which will reflect light into a target plane. We aimed to use partial differential equations along with numerical methods to model the phenomenon of light bouncing off a convex lens.

Motivation

The design of lenses and mirrors, in free form i.e. with no a priori symmetry assumption, has a long list of applications including materials processing, energy concentrators, medicine, antennas, computing lithography, laser weapons, optical data storage, imaging etc. In this project, we reviewed the one dimensional case methods that have been proposed for numerical approximations. We covered LaTeX, Julia, basic numerical analysis, basic linear algebra, and were exposed to the kind of questions researchers address in the numerical analysis of differential equations.

Reflector Equation

If Φ is the luminous flux at x on $[a,b]$, put

$$f(x)=\frac{d\Phi(x)}{dx}$$

for the luminous intensity. The luminous flux of the reflected light is given by $G(\psi)d\psi$. By the conservation of energy,

$$f(x)dx = G(x)d\psi$$

Let $\hat{s}_1$ denote a unit vector in the direction of the parallel incoming light and $\hat{s}_2$ the unit vector in the direction of the reflected light. The unit outward normal to $y=u(x)$ is given by

$$\hat{n}=\frac{\langle p,-1\rangle}{\sqrt{1+p^2}}$$

where $p=u'(x)$. By the vectorial law of reflection,

$$\hat{s}_2=\hat{s}_1-2(\hat{s}_1\cdot\hat{n})\hat{n}$$

Next, we look at

$$d\psi\approx|\hat{s}_2(x+dx)-\hat{s}_2(x)|$$

from which, along with the conservation of energy, it follows that

$$f(x)=G(\psi)\left|\frac{d\hat{s}_2}{dx}\right|$$

noting that $p'=u''(x)$. We take that

$$\frac{d\hat{s}_2}{dx}=\frac{d}{dx}\left(\frac{\langle 2p,-1+p^2\rangle}{1+p^2}\right)$$

$$\implies\left|\frac{d\hat{s}_2}{dx}\right|=\sqrt{\left(\frac{2p'(1-p^2)}{(1+p^2)^2}\right)^2+\left(\frac{4pp'}{(1+p^2)^2}\right)^2}=\frac{2p'}{1+p^2}$$

From here we plug into the flux equation:

$$\Rightarrow G(\psi)\frac{2p'}{1+p^2}=f(x)\Rightarrow\frac{dp}{dx}=\frac{d^2u}{dx^2}=\frac{f(x)(p^2+1)}{2G(\psi)}$$

Afterwards, we can substitute $R(u'(x))=\frac{2G(u'(x))}{((u'(x))^2+1)}$ to get

$$\frac{d^2u}{dx^2}=\frac{f(x)((u'(x))^2+1)}{2G(u'(x))}$$

$$\implies R(u'(x))\cdot u''(x)=f(x)$$

Topics We Covered

- Derivation of the Reflector Equation
- Fixed Point Iteration
- Finite Difference Methods

Fixed Point Iteration (FPI)

The ***fixed points*** of a function, $F(x)$, are those real numbers, x, such that $F(x)=x$.

For example, if we let $F(x)=cos(x)$, for any arbitrary starting x value, if we repeatably apply $F(x)$ to the output, we see the value converges to $x=0.739085...$ which is a point where $cos(x)=x$

However, we are more often than not interested in solutions to $F(x)=0$
Suppose we want to find solutions (0's) to some function

$$F(x)=a_0+a_1x+a_2x^2+...+a_nx^n.$$

This can be done using fixed point iterations by looking for solutions to

$$x=G(x)=\frac{-(a_0+a_2x^2+...+a_nx^n)}{a_1},a_1\neq 0$$

Note: For more complicated functions $F(x)$ we can look at the FPI for $G(x)=F(x)+x$

- If $G(x)$ is continuously differentiable with $G(r)=r$ and $|G'(r)|=S<1$, then the FPI converges to r linearly with rate S for initial guesses sufficiently close to r

Fixed-Point Iteration Example (for a root)

Approximate the root of $x^3-8x-17=0$.

$$x=g(x)=(8x+17)^{\frac{1}{3}}$$

$$g'(x)=\frac{8}{3(8x+17)^{\frac{2}{3}}}$$

Let $x_0=1$. Therefore, we have: $x_{k+1}=(8x_k+17)^{\frac{1}{3}}$, where $x_0=1$. Repeating the preceding iteration 9 times to yields $x_9\approx 3.572$. An online graphing calculator yields the same result. In conclusion $x_{k+1}=(8x_k+17)^{\frac{1}{3}}$, where $x_0=1$ as k approaches ∞, x_k converges to ≈ 3.572

Fixed Point Iteration (for functions of a function)

For this project we are more interested in approximating a function, $u(x)$ starting with an initial guess $u_0(x)$ using fixed point iterations. Yielded from the reflector equation, we have:$u''(x)=\frac{f(x)}{R(u'(x))}$. So writing its iteration in a similar manner to the example presented in the last block we have:

$$u_{k+1}(x)=\frac{f(x)}{R(u_k'(x))}$$

Theorems

Taylor's Theorem For $f(x)$ k-differentiable,

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+...+\frac{h^k}{k!}f^k(c_1)$$

Generalized Intermediate Value Theorem (GIVT) If $f(x)$ cont. on [a,b], $x_1,x_2,...,x_n\in[a,b]$ & $a_1,a_2,...,a_n\geq 0\implies\exists c\in[a,b]$ s.t.

$$(a_1+...+a_n)f(c)=a_1f(x_1)+...+a_nf(x_n)$$

Finite Difference Methods (FDM)

Looking at Taylor's Theorem for k = 4, we arrive at

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2}f''(x)+\frac{h^3}{6}f'''(x)+\frac{h^4}{24}f''''(c_1)$$

$$f(x-h)=f(x)-hf'(x)+\frac{h^2}{2}f''(x)-\frac{h^3}{6}f'''(x)+\frac{h^4}{24}f''''(c_2)$$

with $x-h<c_2<c_1<x+h$.
We can then add these two terms and combine the constants (By the GIVT) to arrive at the **Three-Point Centered-Difference Formula For The Second Derivative**

$$f''(x)=\frac{f(x+h)-2f(x)+f(x-h)}{h^2}-\frac{h^2}{12}f''''(c)$$

where we take the last term as an error term.

Linear BVP's

An example of the applications of Finite Difference Methods is as follows:

$$\begin{cases}u''(x)=F(x)=\beta u(x) & ;0<x<1\\ u(0)=\alpha\\ u(1)=\alpha\end{cases}$$

for $\beta,\alpha\in\mathbb{R}$ given. This can be approximated using FDM. By letting $u(x-h)=w_{i-1}$, $u(x)=w_i$, $u(x+h)=w_{i+1}$ and rearranging the Three-Point Formula for the Second Derivative,

$$\beta w_i=\frac{w_{i-1}-2w_i+w_{i+1}}{h^2}\implies w_{i-1}+(-\beta h^2-2)w_i+w_{i+1}=0$$

with $w_0=\alpha=w_n$. In turn, this can be used to from a system of equations which approximates u(x) on (0,1), with increasing accuracy as we decrease the value of h, subsequently increasing the value of n

$$\begin{cases}w_0+(-\beta h^2-2)w_1+w_2=0\\ w_1+(-\beta h^2-2)w_2+w_3=0\\ \quad ...\\ w_{n-2}+(-\beta h^2-2)w_{n-1}+w_n=0\end{cases}$$

Substituting in α for w_0,w_n and solving the subsequent system of equations (through use of an augmented matrix, etc.)allows us to find values for $w_1,...,w_{n-1}$, which approximates the value to the solution to the BVP at each interval of size h.
Note: For a BVP of form

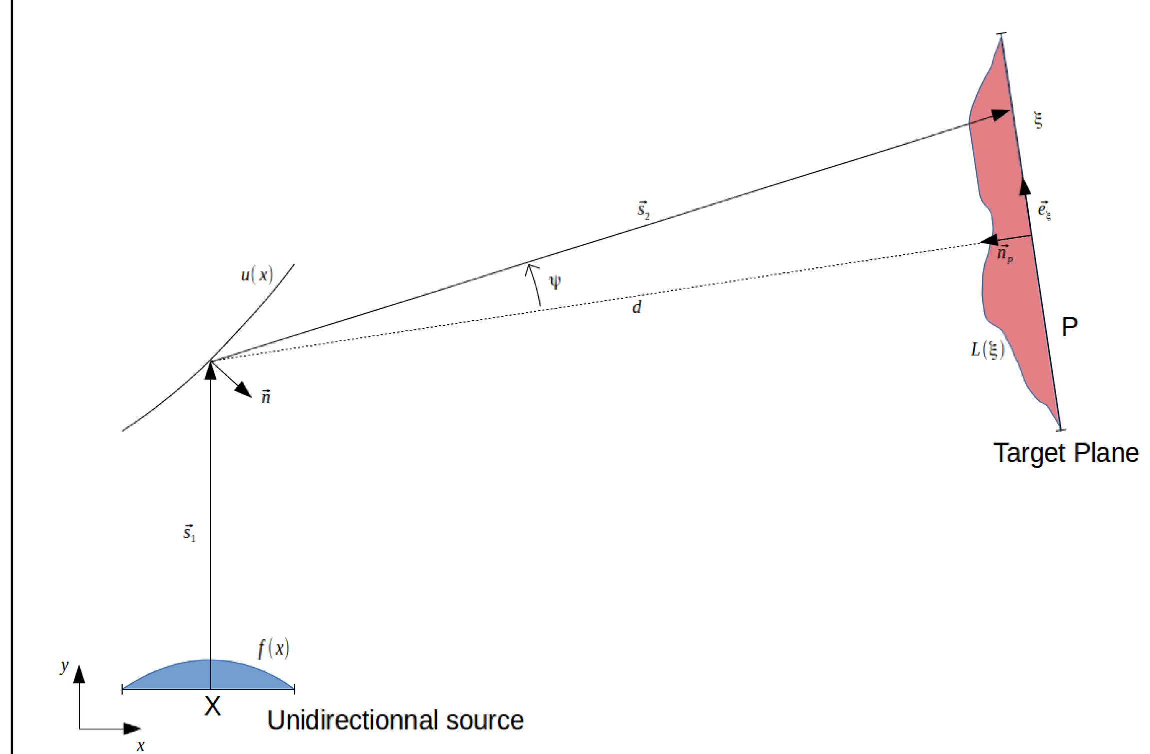
$$\begin{cases}u''(x)=F(x)=\beta u(x) & ;0<x<1\\ u'(0)=\alpha\end{cases}$$

we can first approximate $u'(x)$ using a similar method to find $u'(x+h)\approx u'(x)+hF(x)$ which then give approximations for $u(x)$ through $u(x+h)\approx u(x)+hu'(x)$ for an arbitrary $u(0)\in\mathbb{R}$

Future Questions

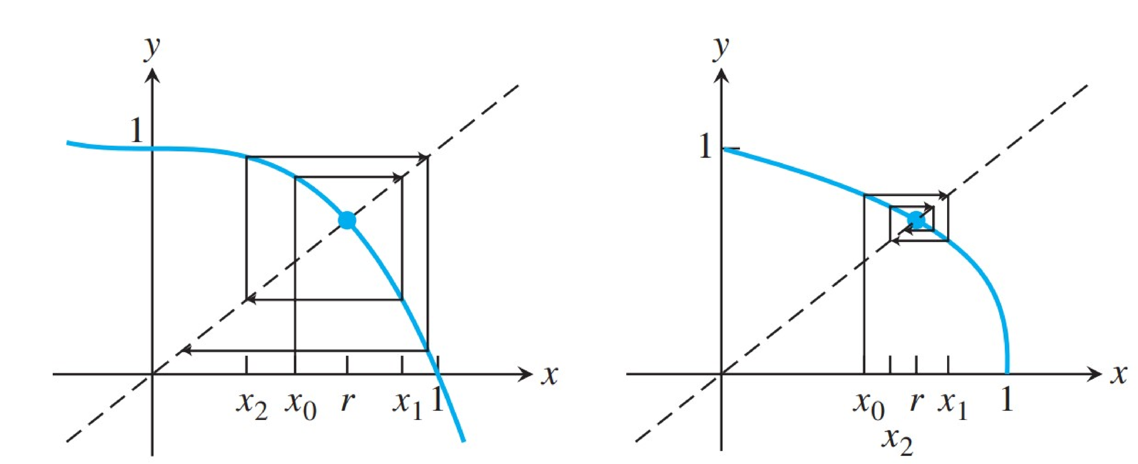
- What if the reflector u(x) is concave rather than convex?
- How do we choose our initial guess, u_0, such that $u_{k+1}=\frac{f(x)}{R(u_k')}$ converges to $u''(x)$ as k approaches ∞?

One-Dimensional Model



Source: https://project.inria.fr/mokabajour/how-it-works

Fixed Point Iteration Example



(Left: $G(x)=1-x^3$; Right: $G(x)=(1-x)^{1/3}$)
For both equations, $G(r)=r$
This serves to show how the condition $|G'(r)|=S<1$ is important to the success of Fixed Point Iteration.
Source: Sauer, T. (2012). Numerical Analysis:Second Edition. Pearson.

Project supervisor

Gerard Awanou

References

https://project.inria.fr/mokabajour/how-it-works

CR Prins. Inverse methods for illumination optics. Ph. D. thesis, 2014.

Sauer, T. (2012). Numerical Analysis:Second Edition. Pearson.