Computing an Invariant of Singularities of Semigroup Rings

Leila Dahlia Kim De Souza John Gut
University of Illinois at Chicago

MSCS Undergraduate Research Laboratory

UIC

Summary

We approached computations of dual F-signatures of affine semigroup rings. The dual F-signature is a numerical invariant of singularities in positive characteristic commutative algebra, and a toric description has recently been given. We used the open source software `polymake` to assist in determining this invariant the semigroup rings given by various rational polyhedral cones over polytopes with varying geometric properties. In particular, we investigated the conjectured formula for the dual F-signature of the Veronese subrings of polynomial rings over fields of prime characteristic $p > 0$.

Specifying an Affine Semigroup Ring via a Rational Polyhedral Cone

A lattice is a free abelian group of finite rank, and can always be thought of as $\mathbb{Z}^d \subseteq \mathbb{R}^d$. We list here two ways to specify a rational polyhedral cone $\sigma \subseteq \mathbb{R}^d$.

Method 1: Using generators

- Choose vectors $\vec{v}_1, \ldots, \vec{v}_r \in \mathbb{Q}^d$.
- The cone $\sigma \subseteq \mathbb{R}^d$ is then given by

$$\sigma = \left\{ \vec{v} \in \mathbb{R}^d \mid \vec{v} = \sum_{i=1}^{r} \alpha_i \vec{v}_i \text{ for some } \alpha_1, \ldots, \alpha_r \in \mathbb{R}_{\geq 0} \right\}.$$

- This method makes it easier to pick out the "edges" (extremal rays) of the cone σ.

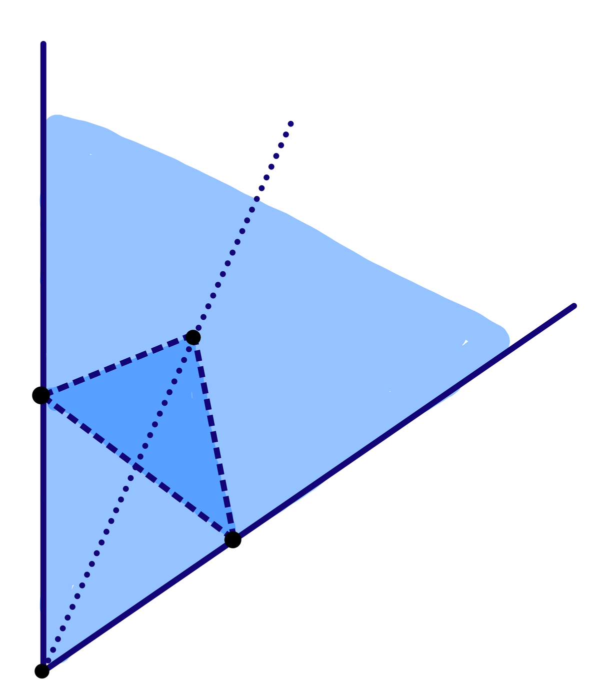
Method 2: Using a rational polytope

- Choose a polytope $P \subseteq \mathbb{R}^d$ where the vertices are in $\mathbb{Z}^d$ or $\mathbb{Q}^d$.
- The cone σ is then given by the set of all rays though the points of P, namely

$$\sigma = \left\{ \vec{v} \in \mathbb{R}^d \mid \vec{v} = \alpha \vec{w} \text{ for some } \vec{w} \in P \text{ and } \alpha \in \mathbb{R}_{\geq 0} \right\}.$$

- This method is particularly useful to specify and visualize a cone in higher dimensions. For example, one can picture a cone in dimension three by specifying a two dimensional "slice."

Once the cone σ has been specified, observe that $\sigma \cap \mathbb{Z}^d$ is a (saturated) sub-semigroup of $\mathbb{N}^d$ – it contains the origin and is closed under sums. The ring associated to the cone σ over a field k is $k[\sigma \cap \mathbb{Z}^d]$ and is the subring R of the Laurent ring $k[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ generated by all of the monomials whose exponent vectors are in σ.



Defining an Invariant

The dual F-signature $s_{\text{dual}}(R)$ is a numerical invariant of singularities first introduced by Sannai. Recently, Tucker and Smirnov have given a formula for computing the dual F-signature of an affine semigroup ring over a field k of prime characteristic $p > 0$.

Theorem

Suppose $\sigma \subseteq \mathbb{R}^d$ is a pointed rational polyhedral cone, and let $R = k[\sigma \cap \mathbb{Z}^d]$ denote the corresponding affine semigroup ring. The dual F-signature of R is expressed as

$$s_{\text{dual}}(R) = \inf_{S} \frac{1}{|S|} \operatorname{vol}\left(\bigcup \{\sigma \cap (\vec{u} - \sigma) \mid \vec{u} \in S\} \right),$$

where the minimum ranges through all finite non-empty subsets S of points $\vec{u} \in \sigma^\circ$. Moreover, we may restrict to subsets S consisting of points $\vec{u} \in \sigma^\circ$ such that $\vec{u} - \vec{m} \notin \sigma^\circ$ for all $0 \neq \vec{m} \in \sigma \cap \mathbb{Z}^d$.

In a specific example, this result gives the following rough algorithm to compute the dual F-signature.

- Specify a number of lattice points $\vec{u}_1, \ldots, \vec{u}_s \in \mathbb{Z}^d$ that are on the interior of the cone σ.
- For each $\vec{u}_i$, consider the polytope $P_i = \sigma \cap (\vec{u}_i - \sigma)$.
- Compute the volume of the region

$$P_1 \cup P_2 \cup \cdots \cup P_s$$

and divide it by the number of interior lattice points of $\sigma \cap \mathbb{Z}^d$ in that region.
- Do this for as many possible finite collections of lattice points as you can and look for the smallest value. The Theorem guarantees that the minimum value is achieved by some finite collections of points that are "sufficiently close to the origin" in some precise manner (*i.e.* so that the polytopes P_i never contain any other interior lattice points of σ other than $\vec{u}_i$).

Computational Complexity

One computes the volume of a union $P_1 \cup P_2 \cup \cdots \cup P_s$ of polytopes in the algorithm above using the inclusion-exclusion principle. For instance, with three polytopes P_1, P_2, P_3, we have

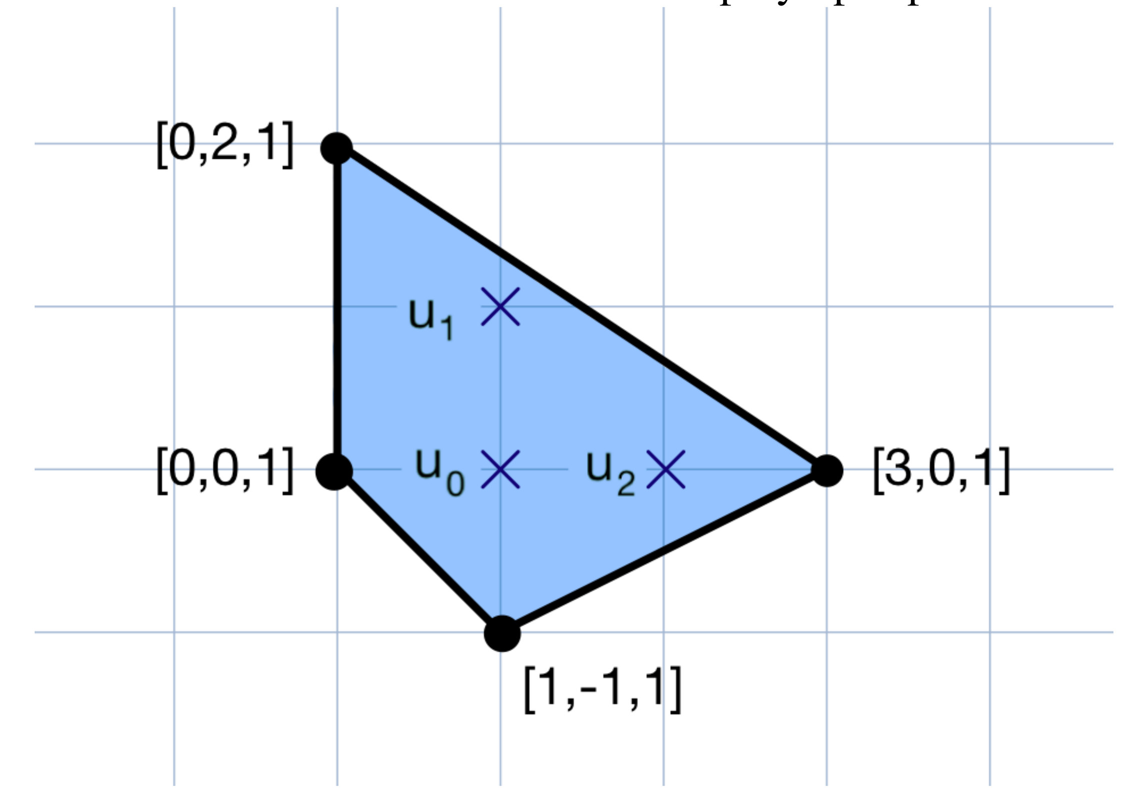
$$\begin{aligned}\operatorname{vol}(P_1 \cup P_2 \cup P_3) = {} & \operatorname{vol}(P_1) + \operatorname{vol}(P_2) + \operatorname{vol}(P_3) \\ & - \operatorname{vol}(P_1 \cap P_2) - \operatorname{vol}(P_1 \cap P_3) - \operatorname{vol}(P_2 \cap P_3) \\ & + \operatorname{vol}(P_1 \cap P_2 \cap P_3).\end{aligned}$$

This is computationally very expensive, and in practice as the number of polytopes increases it becomes practially impossible to do either by hand or using the computer.

In addition, one must also perform the above calculations for all subsets of the interior lattice points of σ that are close to the origin. If the number of interior lattice points of σ that are close is ℓ, the number of non-empty subsets to be checked is $2^\ell - 1$. If ℓ is large, this again leads to more computations than can practically be checked even when using the computer.

Example 1

Suppose $\sigma \subseteq \mathbb{R}^3$ is the strongly convex rational polyhedral cone with rays through the points $[0,0,1], [0,2,1], [3,0,1]$ and $[1,-1,1]$. In other words, σ is the 3-dimensional cone over the polytope pictured below



in the $z = 1$ plane. In this case, the dual F-signature can be computed using only the three interior lattice points $\vec{u}_0 = [1,0,1]$, $\vec{u}_1 = [1,1,1]$, and $\vec{u}_2 = [2,0,1]$ closest to the origin. If we label the corresponding polytopes $P_i = \sigma \cap (\vec{u}_i - \sigma)$, then we can use `polymake` to compute

$\operatorname{vol}(P_0) = 136/441$	$\operatorname{vol}(P_0 \cup P_1)/2 = 187/882$
$\operatorname{vol}(P_1) = 167/882$	$\operatorname{vol}(P_0 \cup P_2)/2 = 89/441$
$\operatorname{vol}(P_2) = 80/441$	$\operatorname{vol}(P_1 \cup P_2)/2 = 571/3528$
	$\operatorname{vol}(P_1 \cup P_2 \cup P_3)/3 = 101/588$

and we check that the minimum achieved is $571/3528$ which gives the dual F-signature.

Example 2: Veronese Subrings of Polynomial Rings

The degree n Veronese subring of a polynomial ring with d variables can be described as $k[\ldots, x_1^{a_1} \ldots x_d^{a_d}, \ldots]$ where the set of generators varies over all the monomials with $a_1 + \ldots + a_d = n$. For example, in two variables this can be denoted by $k[x^n, x^{n-1}y, \ldots, xy^{n-1}] \subseteq k[x,y]$. While not immediately clear from the definition, each of these is also an affine semigroup as decribed above. For example, when the number of variables $d = 3$, one may consider the degree n Veronese subring to be determined by the cone with generators $[0,0,1], [n,0,1], [0,n,1]$. Similar expressions are valid in higher dimensions as well. Smirnov and Tucker have made the following conjecture for the dual F-signatures of Veronese subrings of polynomial rings.

Conjecture

If $R(n,d)$ is the degree n Veronese subring of a polynomial ring in d variables, then $s_{\text{dual}}(R(n,d)) = \lceil n/d \rceil \cdot 1/n$. In particular, if $d \geq n$ then $s_{\text{dual}}(R(n,d)) = 1/n$.

We used `polymake` to run experiments and verify the conjecture when $d = 3$ and $n \leq 6$.

Example 3

In many cases, we found that writing down a "random" cone – even in low dimensions – led to an example where we were unable to fully calculate the dual F-signature.

For example, we attempted to compute the F-signature of the cone generated by the lattice points

$$[1,0,0,0], [1,1,0,2], [1,1,0,6], [1,1,5,2], [1,1,4,-4]$$

and found that our code did not terminate, even after allowing it to run for over 36 hours.

Conclusions

We found that `polymake` can be a highly effective tool for computations in polyhedral geometry. Its promise as a research tool is also clear through its implementation into OSCAR and other computational libraries. In particular, it can be used to explore invariants such as the dual F-signature of affine semigroup rings, and allow one to make and test formulas such as the conjectured value of the dual F-signature of Veronese subrings. However, the computational complexity of actually computing the dual F-signature may prove to be prohibitive in practice in some cases, and Example 3 demonstrates this barrier where we found that our hardware was insufficient.

References

Ilya Smirnov and Kevin Tucker.
Towards the theory of F-rational signature, 2019.
arXiv:1911.02642.

H. Li and F. Van Oystaeyen.
A Primer of Algebraic Geometry: Constructive Computational Methods.
Chapman & Hall/CRC Pure and Applied Mathematics. Taylor & Francis, 2000.

Akiyoshi Sannai.
On Dual F-Signature.
International Mathematics Research Notices, 2015(1):197–211, 09 2013.

William Fulton.
Introduction to Toric Varieties.
Annals of Mathematics Studies: Princeton University of Press, Princeton, New Jersey, 1993.

Ewgenij Gawrilow and Michael Joswig.
`polymake`: a framework for analyzing convex polytopes.
In *Polytopes—combinatorics and computation (Oberwolfach, 1997)*, volume 29 of *DMV Sem.*, pages 43–73. Birkhäuser, Basel, 2000.