Arboreal Dynamics and Symmetries in the Roots of Polynomials

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Summary

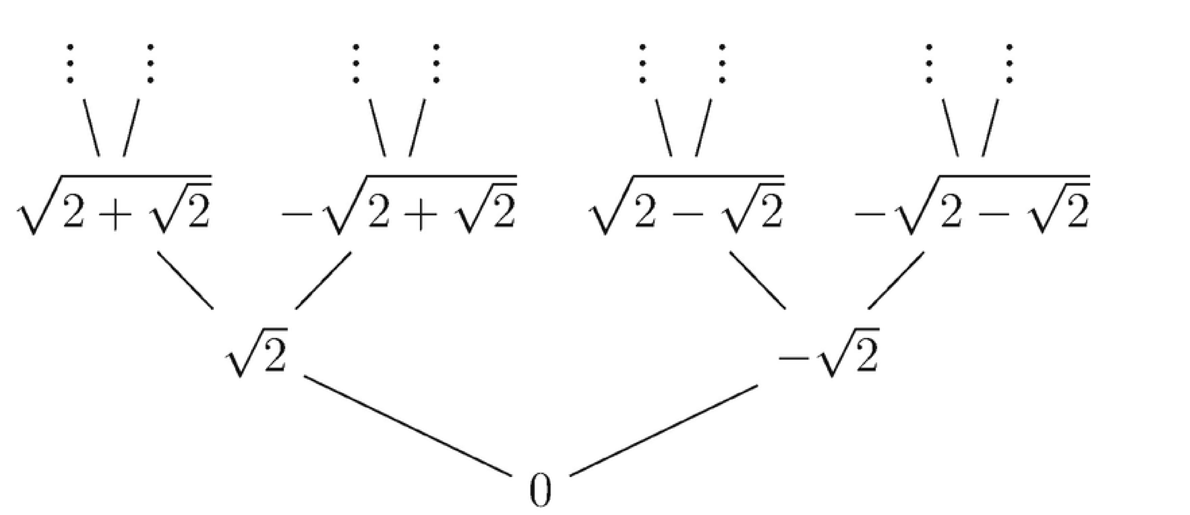
Let K be a number field (which we can assume to be $\mathbb{Q}$) with algebraic closure $\overline{K}$, and let f be a polynomial over K with degree $d \geq 2$. Denote f^n as n-fold composition of f with itself, fix $\alpha \in K$ then the n-th inverse image of α is the set $f^{-n}(\alpha) = \{\beta \in \overline{K} \mid f^n(\beta) = \alpha\}$ For all $n \geq 1$, let K_n be the field generated by $f^{-n}(\alpha)$ then it is known that we also obtained a tower of Galois field extension $K = K_0 \subseteq K_1 \subseteq K_2 \subseteq ....$ We are interested in the cases where K_n is an abelian extension for all n, that is, $G_n = Gal(K_n/K)$ is an abelian group. In particular, we are interested in the rank or the number of generators of G_n for all n in such cases.

(Andrews-Petsche) Conjecture for K_n to be an abelian extension for all $n \geq 1$

1) $f(x) = x^d$ and $\alpha = \zeta$ where ζ is a root of unity

2) $f(x) = T_d(x)$, which is the d-th Chebyshev polynomial and $\alpha = \zeta + \zeta^{-1}$ where ζ is a root of unity

$Gal(K_n/K)$ acts faithfully on the n-th preimage tree



Why abelian?

It is known that for polynomial f with degree d, we have:

$$d^n \leq |Gal(K_n/K)| \leq d!^{(d^n-1)/(d-1)}$$

If K_n/K is an abelian extension then we achieve the lower bound, meaning $|Gal(K_n/K)| = d^n$

Chebyshev Polynomials

Explain why Chebyshev polynomials are useful in investigating these questions

Chebyshev polynomials of the first kind T_n are given by:

$$T_n(\cos\theta) = \cos(n\theta).$$

The first Chebyshev polynomials of the first kind are

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$T_4(x) = 8x^4 - 8x^2 + 1$$
$$T_5(x) = 16x^5 - 20x^3 + 5x$$

Field extension

Let F be a field then a field L is an extension field of F if F is (isomorphic to) a subfield of L Equivalently, L is an extension field of F if it is both a field and a vector space over F

The field L is a finite extension of F (of degree n) if it is a finite (n-)dimensional vector space over F

Aut(L/F)

Let $F \subset L$ be a finite extension. Then

$$Aut(L/F) := \{\sigma : L \to L : \sigma \text{ is an automorphism}, \sigma(a) = a \; \forall a \in F\}$$

The Galois extensions

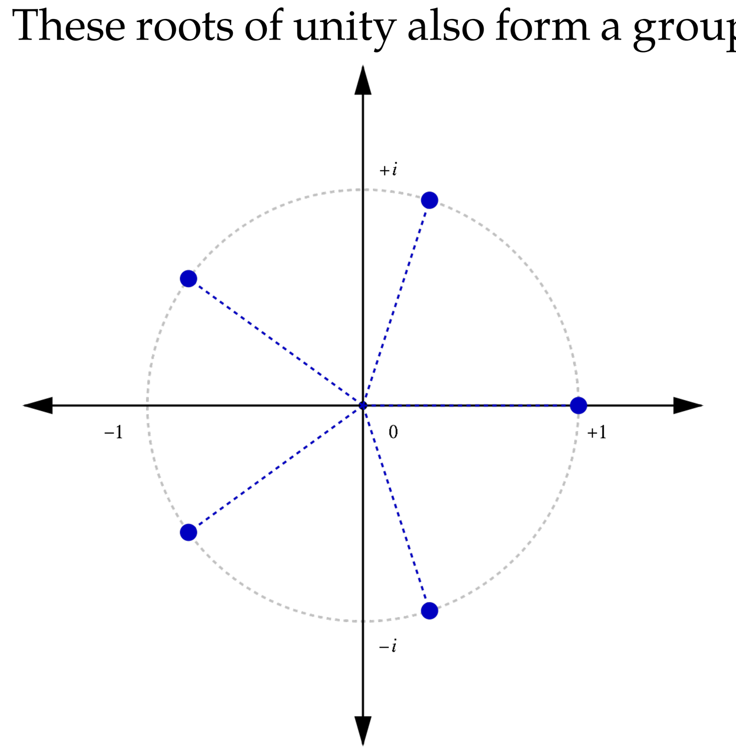
A finite field extension $F \subseteq L$ of degree n is a Galois extension of F if $|Aut(L/F)| = n$

In such cases, $Aut(L/F)$ is denoted $Gal(L/F)$.

Roots of unity

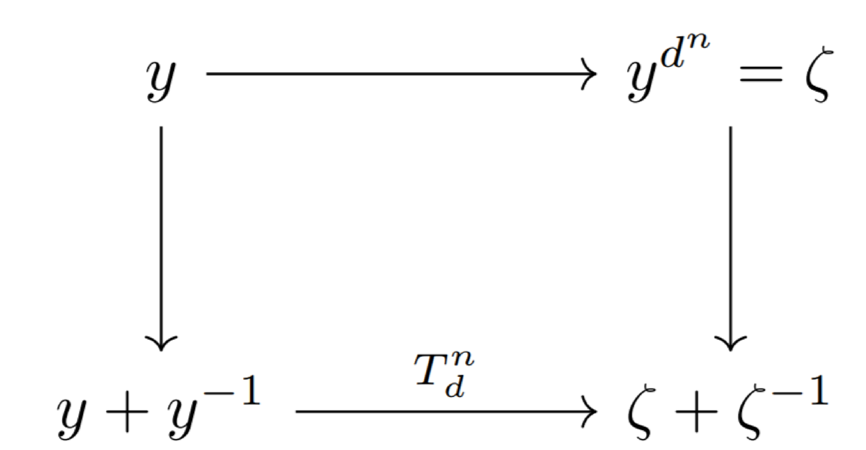
The set of solution to the equation $x^n = 1$ is called the (n-th) roots of unity.

These roots of unity also form a group under multiplication



An interesting square for Chebyshev polynomials

Let ζ be a r-th root of unity then all solutions to the equation $T_d^n(x) = \zeta + \zeta^{-1}$ are of the form $y + y^{-1}$ where $y^{d^n} = \zeta$ which means y is rd^n-th root of unity and gives rise to a diagram of correspondence:



Assuming $K = \mathbb{Q}$, then the field $K_n = \mathbb{Q}(y + y^{-1} \mid y^{d^n} = \zeta)$

Questions

1) We know that if the extension K_n/K is abelian then $Gal(K_n/K)$ is at minimal size (see: Why abelian?), but is the converse true?

Equivalentiy, is there a polynomial f with degree $d \geq 2$ such that $|Gal(K_n/K)| = d^n$ but $Gal(K_n/K)$ not abelian for all but finitely many n?

2) Is the conjecture for K_n/K being abelian extension for all $n \geq 1$ true?

**The conjecture holds for $f(x) = x^2$ so far

3) Assuming K_n/K is an abelian extension for all n, what are the ranks (number of generators) and growth rate of the ranks of $Gal(K_n/K)$?

Algorithms We Used

In order to find the following data, we set $\zeta = e^{\frac{2\pi i}{r}}$, we then found the rank of the Galois Group of $T_2^n(x) = \zeta + \zeta^{-1}$ for various values of n and r.

We were able to optimize this calculation by exploiting the "square" relationship presented above. It allows us to deduce that the solutions to the above equation are all of the form $2cos(\frac{2\pi(1+rl)}{r*2n})$ when $l = 0, 1, ..., d^n - 1$.

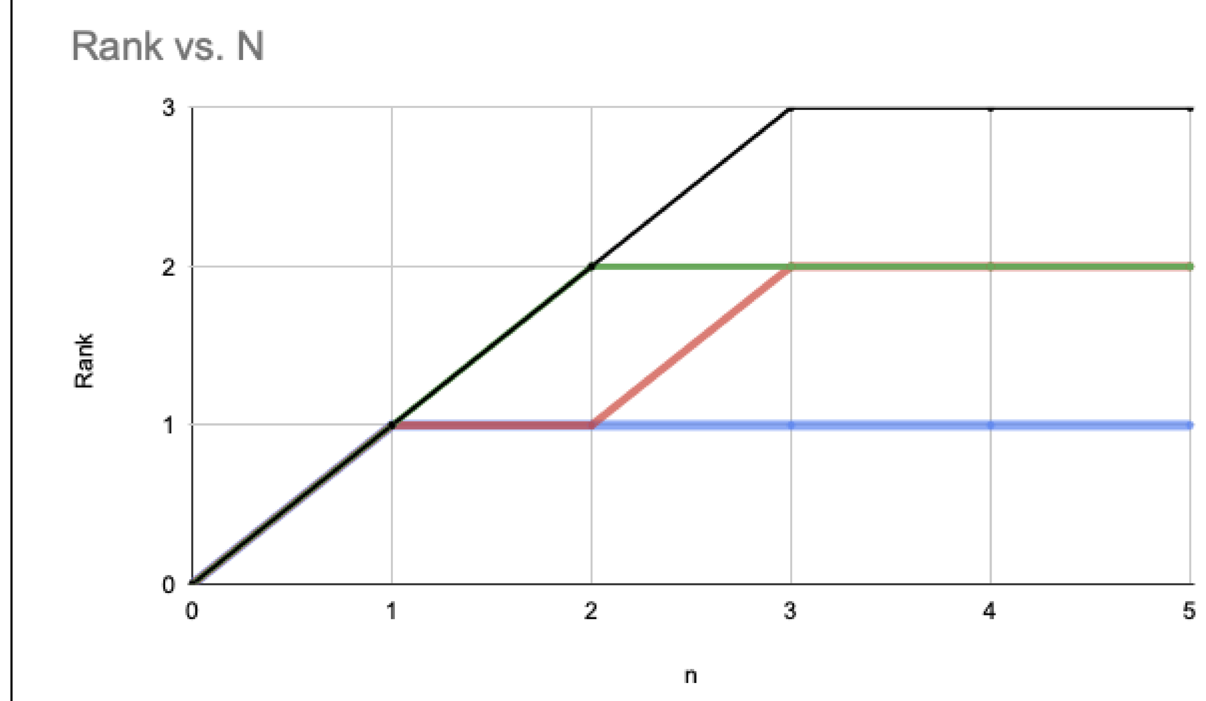
Using this information we could find the splitting field by calculating the minimum polynomial of the list of solutions and then getting the Galois Group of that field.

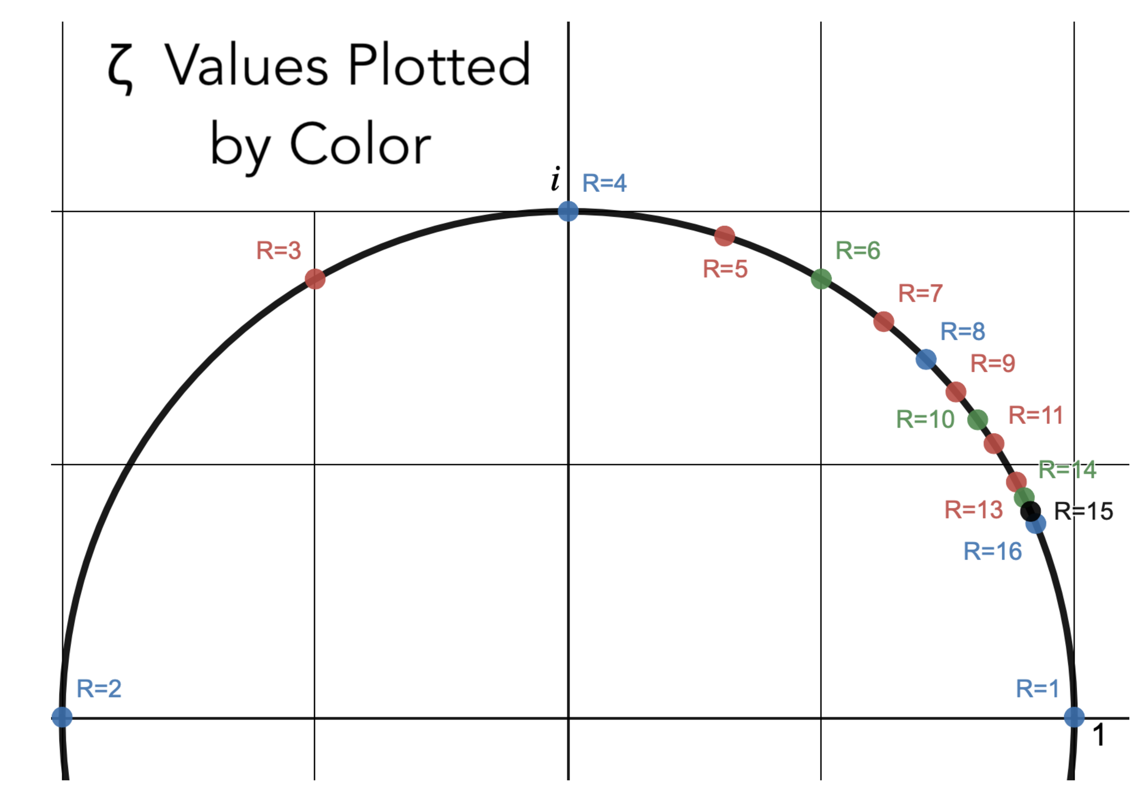
However this is still an exponential time algorithm so we had difficulties scaling for larger values of n.

Experiments

The graph below shows the plot n against the rank of the the Galois Group of $T_2^n(x) = \zeta + \zeta^{-1}$ for various values of $\zeta = e^{\frac{2\pi i}{r}}$. When $r \leq 16$ there seem to be 5 different paths the function will take, 4 of which are shown below.

Below that we have plotted the values on the complex plane and matched them to the graph by color.





Conclusions

Looking at the data, it is interesting to note that all the r values of the form 2^k follow the blue line, all the r values that are primes or prime powers follow the red line, and all the r values that are prime or prime powers multiplied by 2 follow the green line. We have seen this pattern carry on up until $r = 32$.

So although it is not obvious what the relationship is between the rank of the Galois Group and the primality of r, it is clear that there is a correlation between the 2.

If we were able to collect more data for higher values of both n and r we could make this claim more precise.

References

J. Andrews, C. Petsche. *Abelian Extensions in Dynamical Galois Theory*. (2021).

D. Cox. *Galois Theory 2nd ed.*. John Wiley & Sons (2012), 73–144.