Finding Roots of Matrix Polynomials

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Introduction

- We try to solve matrix polynomials and analyze the behavior of the roots.
- Each entry of a matrix is a polynomial, and each polynomial is solved for its roots.
- The length of the given word determines the power of each polynomial.
- Using a software (PHCpack) that takes such matrices as input, we find the roots.
- We calculate the roots of $P(x) = z$ as z moves around a loop.
- Then, we compute the distance each root travels to record its path.
- After a loop is complete, we classify the path the roots have taken.

Motivation

We want to show that there are permutations that roots of polynomials make as the image traverses a loop, in relation to Galois theory.

- Let $f(z)$, $z \in \mathbb{C}$ be a polynomial.
- $z_0 \in \mathbb{C}$ generic $\Rightarrow f(z) = z_0$ has $d = \deg(f)$ roots R_0.
- $z_0 = z_0(t)$ traverses a loop $\alpha \rightsquigarrow$ permutation $\sigma(\alpha)$ of R_0.
- Galois group of (splitting field of) f: all permutations arise in this way.

Related Notions

- Notion of Galois theory over matrix rings is due to Jacobson and Cartan.
- Nonsolvability of matrix polynomials.

Generalization and Definitions

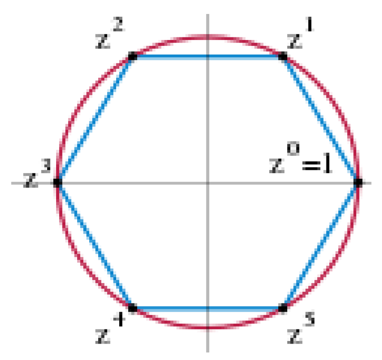
Matrix polynomials are polynomials where the variables are matrices. The expression x^2+2 would be in matrix polynomials akin to xxy, where x is a variable matrix and y is a matrix of constants. These expressions are also known as **words**. Each word represents a 2 by 2 matrix, which belongs to the special linear group with its determinant equal to 1.

Main Question

Thus the question becomes: Are there any non-trivial automorphisms for matrix polynomials? Using methods for solving these systems, we animated, outputted, and analyzed the paths of roots of matrix polynomials to answer this question.

Theory

Solving systems of polynomial equations is an important subject in various areas in mathematics. Finding real and complex roots of quadratic polynomials with rational coefficients only takes a quadratic formula to find them. The quadratic formula contains only addition, subtraction, multiplication, and division. Thus, a quadratic polynomial is said to be *solvable* by radicals. The task of solving polynomials becomes more complicated as the degree of a polynomial is greater than 4, because then no such formulas exist. In order to find roots of higher degree polynomials, instead of finding the exact roots, Galois suggested studying the relationships and symmetries between their roots. For example, given a polynomial of degree n, the polynomial has exactly n roots (including complex roots), and the roots are symmetric. This relationship makes their rearrangement possible in both the real and complex cases. The collection of symmetries in a polynomial's Galois group can be used to determine whether or not the group is solvable. For example, a *cyclic* group is one that is generated by a single element. A cyclic group is also *abelian*, where the order of operations does not affect the result. Any abelian group is also solvable. Below is an example of such group:



Polynomials need not have rational coefficients. For example, a matrix can be a coefficient of a polynomial. In the case of invertible matrices, the order of operations matters, so the polynomials form a non-commutative ring. For matrix polynomials, root finding must follow steps used for a non-abelian group. Matrices with polynomial entries run into the same solvability problem as regular polynomials once the degree exceeds 4. There now exist algorithms that effectively estimate roots to such non-linear systems, such as the *homotopy continuation method*. Note that this method estimates the solutions since multivariate systems of polynomials generally cannot be solved exactly.

Studying the roots of such polynomials using the fore-mentioned method can manifest both trivial and non-trivial behaviors of the roots. A non-trivial behavior is classified as one where a root swaps places with other roots in accordance with an adjustment of parameters for the entries. For instance, consider roots A, B, C and D. A non-trivial automorphism would result of the roots taking place of each other; as an example, A would take place of B, B would take place of C, C would take place of D, and D would take place of A. This permutation would thus be denoted (A B C D). It could be the case that the roots swap only as pairs, where A swaps with B, B swaps with A, C swaps with D, and D swaps with C. In this case, the permutation would be written as (A B) (C D). A permutation would be considered trivial if the roots did not swap with each other.

Result

We developed a program that solves matrix polynomial equations, and we found that there exist matrix polynomials with nontrivial Galois group.

How It Works

Let $w(x,y)$ be a word in 2 variables x, y and their inverses. Fix $z_0 \in SL(2,\mathbb{R}), y_0 \in SL(2,\mathbb{R}), x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$

$$z_0 \& y_0 := \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \Psi & 0 \\ 0 & 1/\Psi \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

$$e.g.\ xxy_0 = z_0$$

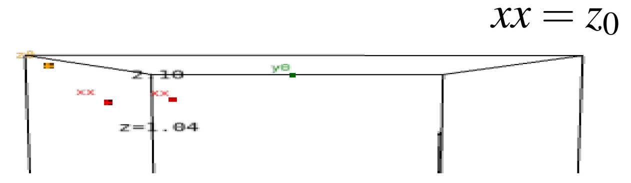
- PHCpack finds the roots of the linear equations.
- Real roots are decomposed using SVD-decomposition and are then plotted.
- Loops in text form are matched using distance formula.
- To find the loops, the parameters of the SVD composition of the words are varied.
- $0 < \theta_1 < \pi$
- $0 < \theta_2 < 2\pi$
- $0 < \Psi < \infty$

Experiments

- Find Galois Automorphisms
- Graph real roots

First

$$xx = z_0$$



- Fetch with real and imaginary roots

Second

$$\delta = 0.1, 0.1 <= \theta_2 < 3.1$$

```
xx
[([1, 1],), ([2, 2],), ([3, 3],), ([4, 4],)]
```

(11)(22)(33)(44)

Third

$$xy_0x = z_0$$

$$\delta = 0.1, 0¡= \theta_2 < 6.2$$

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xyx
[([1, 1], [2, 1], [3, 1]), ([2, 2], [1, 2], [4, 2]), ([3, 3],), ([4, 4],)]
```

$$(11)(22)(34)(43)$$

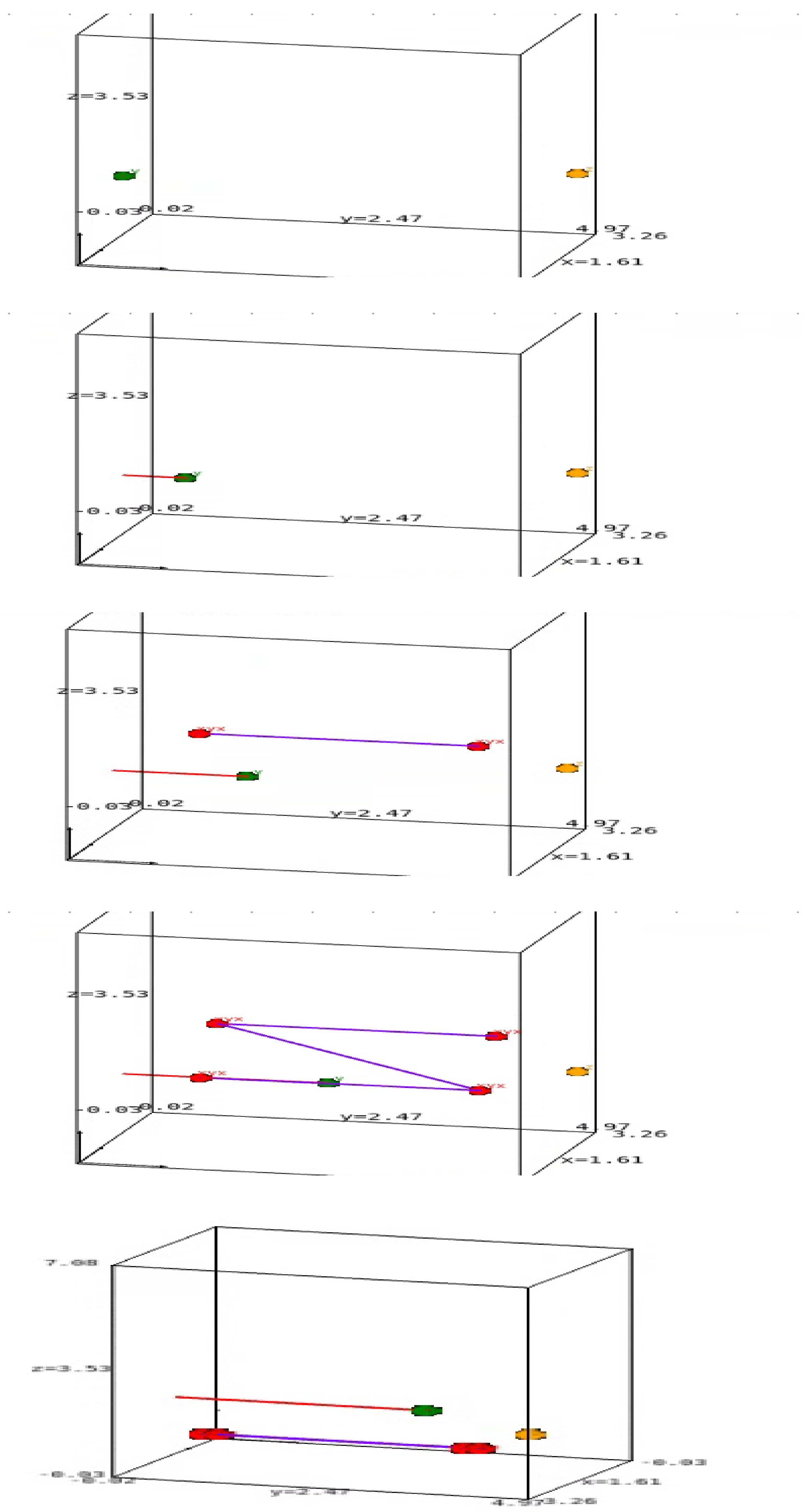
$$(12)(21)(34)(43)$$

What's Next?

- Finding the Galois group.
- Looking at square matrices of $n^2, n > 2$.
- Polynomials of multiple terms e.g.

$$xxy_0x + xx + xy_0x = 0$$

Still Frames of Loop $xyx\ \delta_{\theta_2}$



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[Illustration]. www.pngwing.com/en/search?q=abelian+Group